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Discretizing of linear systems with time-delay Using method of Euler's and Tustin's approximations

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ABSTRACT
Delays deteriorate the control performance and could destabilize the overall system in the theory of discrete-time signals and dynamic systems. Whenever a computer is used in measurement, signal processing or control applications, the data as seen from the computer and systems involved are naturally discrete-time because a computer executes program code at discrete points of time. Theory of discrete-time dynamic signals and systems is useful in design and analysis of control systems, signal filters, state estimators and model estimation from time-series of process data system identification. In this paper, a new approximated discretization method and digital design for control systems with delays is proposed. System is transformed to a discrete-time model with time delays. To implement the digital modeling, we used the z-transfer functions matrix which is a useful model type of discrete-time systems, being analogous to the Laplace-transform for continuous-time systems. The most important use of the z-transform is for defining z-transfer functions matrix is employed to obtain an extended discrete-time. The proposed method can closely approximate the step response of the original continuous time-delayed control system by choosing various of energy loss level. Illustrative example is simulated to demonstrate the effectiveness of the developed method.
***Keywords* -** Discretization, Time-delay systems, Z-transform.

I. INTRODUCTION

Sampling a continuous-time system is a fundamental problem in a variety of scientific areas, such as computer control, system identification and signal processing. It is becoming even more conspicuous in light of the huge success of computer-aided processing and networking [9, 12]. There are many intriguing problems related to sampling. Time delay is one of the key factors influencing the overall system stability and performance. In particular, as the different effects of actuator, sensor and controller exist in control systems, delays are often formulated as state time delays, input time delays as well as output time delays in a continuous-time or discrete-time framework [8, 14]. To digitally simulate and design a continuous-time delayed control system, it is often required to obtain an equivalent discrete-time model. The digital modeling of continuous-time systems with input delays can be found in a standard textbook .To improve the performance of a continuous-time system with multiple time delays, several advanced control theories and practical design techniques have been proposed [7,10]. Most control systems are formulated in a continuous-time framework, for which many analysis tools and control methodologies are well-established. With the rapid advances in digital technology and computers, digital control provides various advantages over its analog counterpart for better reliability, lower cost, smaller size, more flexibility and better performance. The resulting digitally controlled continuous time system becomes a sampled-data system.

Systems including time delay, due to the system dynamics, are widely present in the industry which imposes a lot of constraints that make the control and computer programming of such system difficult [1].Then the control of a system with a time delay is generally difficult; due to the constraints imposed by the time delay. These constraints can cause performance deterioration that leads the process to instability especially when operating in closed loop [2, 3, 4, 11].Most physical systems, a macroscopic point of view, are continuous. In modern control systems, information is digitally processed which requires sampling signals [3, 5]. One speaks in this case of sampled or discrete systems. If necessary, it is always possible to obtain the state equations from the transfer-function matrix [13]. Another advantage of using the transfer-function matrix is that one can decompose the multivariable system into p subsystems, each with one output and m inputs. For this reason we need the discretization of continuous time-delay systems.

The objective of this paper is to extend and analyze the ideas in the just cited references; we consider three methods for obtaining the discrete-time delay approximation. These are:

(i) Backward difference method.
(ii) Forward difference method.
(iii) Bilinear z-transformation method.

The paper is organized as follows: The next section discusses discretization of systems with external point delay; Section 3 provides numerical examples; Section 4 includes a comparison between the three methods followed by a conclusion in the end of the paper.

II. DISCRETIZATION OF SYSTEMS WITH EXTERNAL POINT DELAY

In a digital computer, time cannot flow continuously as it is perceived in the physical world. The time is defined on a discrete set of times, which are separated by a regular time interval known by one sampling period. It is therefore necessary to define new mathematical tools adapted to discrete time, to represent the sampled signals and systems and to adapt tools and methods for automatic analog continuous time in the design of digital controllers.

Then our problem may be stated as the determination of a discrete time approximation corresponding to the following state-space equations:

$$\begin{cases} \dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Bu(t) \\ y(t) = Cx(t) \end{cases} \quad (1)$$

Where:

$A_0 x(t)$: Original Term state.

$A_1 x(t-h)$: Delayed Term state.

With $h = qT$: a multiple delay the sampling period is an integer q .

T : The sampling period assumed chosen suitably.

x, u *and* y Respectively are the state vector, the vector and the control vector output.

A_0, A_1, B *and* C are matrices of suitable dimensions.

Calculating the Laplace transform of the system (1), we get the following equation model:

$$\begin{cases} pX(p) - X(0) = A_0 X(p) + A_1 e^{-hp} X(p) + BU(p) \\ Y(p) = CX(p) \qquad ; X(0) = 0 \end{cases} \quad (2)$$

This is equivalent to this equation:

$$X(p)\left[pI - A_0 - A_1 e^{-hp}\right] = BU(p) \quad (3)$$

This equation can be written as:

$$X(p) = \left[pI - A_0 - A_1 e^{-hp}\right]^{-1} BU(p) \quad (4)$$

Then:

$$Y(p) = CX(p) = C\left[pI - A_0 - A_1 e^{-hp}\right]^{-1} BU(p) \quad (5)$$

From (5) we can get:

$$H(p) = \frac{Y(p)}{U(p)} = C\left[pI - A_0 - A_1 e^{-hp}\right]^{-1} B \quad (6)$$

Implementation of continuous-time control and filtering functions in a computer program, in some cases we need to find a discrete-time z-transfer function matrix from a given continuous-time p-transfer function. In accurate model based design of a discrete controller for a process originally in the form of a continuous-time p-transfer function, $H(p)$ .The latter should be discredited to get a discrete-time process model before the design is started. There are several methods for discretization of a p-transfer function. The discretization can be realized in several ways by calculating the z-transfer function matrix for $H(p)$ .The methods can be categorized as follows, and they are described in the following section:

A. Forward difference method

This method is also called Forward rectangular rule. Here we apply the technique of the first order approximation by linear transformation. These approximations on the z-transform function matrix exploit the relationship: $z = e^{pT}$.The idea is to approximate this relationship by a linear relationship between z and p.

When the sampling period T is small, the linear approximation of the first order function exponential gives: $z = e^{pT} \approx 1 + pT$, so we can write: $p = \frac{z-1}{T} = \frac{1-z^{-1}}{Tz^{-1}}$ (7)

This technique of discretization results from the approximation of the derivative of two sampling instants:

$$L^{-1}\left(pX(p)\right) = \frac{dx(t)}{dt} \approx \frac{x(t+T)-x(t)}{T} = z^{-1}\left(\frac{z-1}{T}X(z)\right) \quad (8)$$

It can be shown that the discrete-time transfer function matrix of the transfer function matrix given in (6) is obtained by using the Forward difference method after substituting for $p = \frac{z-1}{T}$:

$$H(z) = C\left[\left(\frac{z-1}{T}\right)I - A_0 - A_1 e^{-h\left(\frac{z-1}{T}\right)}\right]^{-1} B \quad (9)$$

From (7) we can write:

$$H(z) = C\left[\left(\frac{z-1}{T}\right)I - A_0 - A_1 e^{-h\left(\frac{z-1}{T}\right)}\right]^{-1} B \quad (10)$$

By applying an approximation Taylor expansion near 0 of the term $e^{-h\left(\frac{z-1}{T}\right)}$ to the first order approximation, we can write:

$$e^{-h\left(\frac{z-1}{T}\right)} = 1 - h\frac{z-1}{T} \quad (11)$$

The expression of $H(z)$ becomes:

$$H(z) = CTz^{-1}\left[I + A_1 h - (I + A_0 T + A_1 T + A_1 h)z^{-1}\right]^{-1} B \quad (12)$$

B. Backward difference method

This method is also called backward rectangular rule. To simulate the entire system as a discrete-time system one needs to find the discrete equivalent. Here we apply the technique of the first order approximation. This Backward discretization results from approximation (8) of the derivative that can be made between two sampling times:

$$L^{-1}\left(pX(p)\right) = \frac{dx(t)}{dt} \approx \frac{x(t)-x(t-T)}{T} = z^{-1}\left(\frac{z-1}{zT}X(z)\right) \quad (13)$$

Then, we can write: $p = \frac{(z-1)}{zT} = \frac{1-z^{-1}}{T}$ (14)

It can be shown that the discrete-time transfer function matrix of the transfer function matrix given in (6) is obtained by using the backward difference method after substituting for $p = \frac{(z-1)}{zT}$:

$$H(z) = C\left[\left(\frac{z-1}{zT}\right)I - A_0 - A_1 e^{-h\left(\frac{z-1}{zT}\right)}\right]^{-1} B \quad (15)$$

By applying an approximation Taylor expansion near 0 of the term $e^{-h\left(\frac{z-1}{zT}\right)}$ to the first order approximation, we can write:

$$e^{-h\left(\frac{z-1}{zT}\right)} = 1 - h\frac{z-1}{zT} \quad (16)$$

The expression of $H(z)$ becomes:

$$H(z) = C\left[\frac{I}{T} - A_0 - A_1 + \frac{A_1 h}{T} - \left(\frac{A_1 h}{T} + \frac{I}{T}\right)z^{-1}\right]^{-1} B \quad (17)$$

C. Bilinear z-transformation method

This method is also called Tustin's method also the trapezoid rule in digital control community, there is a good discrete-time approximation for a continuous-time linear system is obtained through the bilinear z-transformation if the sampling interval T is selected suitably so that $wT \le 0.5$, where w is the magnitude of the pole of the transfer function of the continuous-time system far from the origin of the P-plane [6]. However, while the bilinear z-transformation leads to a realization in the form of a transfer-function matrix, the method based on the trapezoidal rule leads directly to a state space realization, more suitable for digital simulation.

We apply a second-order approximation by bilinear transformation homographic: Tustin discretization. The linear approximation of the second order function exponential gives:

$$z = \frac{1 + P\frac{T}{2}}{1 - P\frac{T}{2}} \quad (18)$$

Therefore: $p = \frac{2}{T}\frac{z-1}{z+1} = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$ (19)

This approximation results from the approximate numerical integration by the trapezoidal rule, between two sampling times, indeed, either:

$$y(t)=\int x(t)dt \quad \rightarrow \quad Y(p)=\frac{1}{p}X(p) \tag{20}$$

From where: $p=\dfrac{X(\mathrm{p})}{Y(p)}$ (21)

By approximating between two seconds sampling, we get:

$$y_k=y(kT)\approx y_{k-1}+\frac{X_{k-1}+X_k}{2}T \tag{22}$$

The z-transform function gives then:

$$\left(1-z^{-1}\right)Y(z)=\frac{T}{2}\left(1+z^{-1}\right)X(z) \tag{23}$$

From where: $\dfrac{X(z)}{Y(z)}=\dfrac{2}{T}\dfrac{z-1}{z+1}$ (24)

Then, we can write:

$$p=\frac{2}{T}\frac{z-1}{z+1}=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \tag{25}$$

It can be shown that the discrete-time transfer function matrix of the transfer function matrix given in (6) is obtained by using the bilinear transformation method after substituting for $p=\dfrac{2}{T}\dfrac{1-z^{-1}}{1+z^{-1}}$:

$$H(z)=C\left[\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}I-A_0-A_1e^{-\frac{2h}{T}\frac{1-z^{-1}}{1+z^{-1}}}\right]^{-1}B \tag{26}$$

By applying an approximation Taylor expansion near 0 of the term $e^{-\frac{2h}{T}\frac{1-z^{-1}}{1+z^{-1}}}$ to the first order approximation, we can write:

$$e^{-\frac{2h}{T}\frac{1-z^{-1}}{1+z^{-1}}}=1-h\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \tag{27}$$

The expression of $H(z)$ becomes:

$$H(z)=C(1+z^{-1})\left[I+A_1h-A_0\frac{T}{2}-A_1\frac{T}{2}-(I-A_1h+A_0\frac{T}{2}+A_1\frac{T}{2})z^{-1}\right]^{-1}B\frac{T}{2} \tag{28}$$

III. NUMERICAL EXAMPLES

In this section an example of application of the proposed approach are presented. This example shows the validity of those methodologies. As the system is considered by this equation:

$$\begin{cases}\dot{x}(t)=\begin{pmatrix}0 & 1\\ -2 & -3\end{pmatrix}x(t)+\begin{pmatrix}0 & 0\\ -1 & 2\end{pmatrix}x(t-0.2)+\begin{pmatrix}0\\ 1\end{pmatrix}u(t)\\ y(t)=(1\ \ 1)\,x(t)\end{cases} \tag{29}$$

In this section we are treating an example of a continuous linear system. A delay is only in the state which was chosen a sampling period T=0.2s and applying the Forward difference method, the following discrete time-delay system is obtained:

$$H(z)=CTz^{-1}\left[I+A_1h-(I+A_0T+A_1T+A_1h)z^{-1}\right]^{-1}B \tag{30}$$

With:

$A_0=\begin{pmatrix}0 & 1\\ -2 & -3\end{pmatrix}$; $A_1=\begin{pmatrix}0 & 0\\ -1 & 2\end{pmatrix}$;

$B=\begin{pmatrix}0\\ 1\end{pmatrix}$; $C=\begin{pmatrix}1 & 1\end{pmatrix}$ *and* $h=qT=0.2s$

The operator z^{-1} is here a time-step delay operator, and it can be regarded as an operator of the time-step delay, z^{-1} is also the transfer function of a time-step delay.

The simulation result in the step responses of the continuous system and approximate discrete system is depicted in figure1.

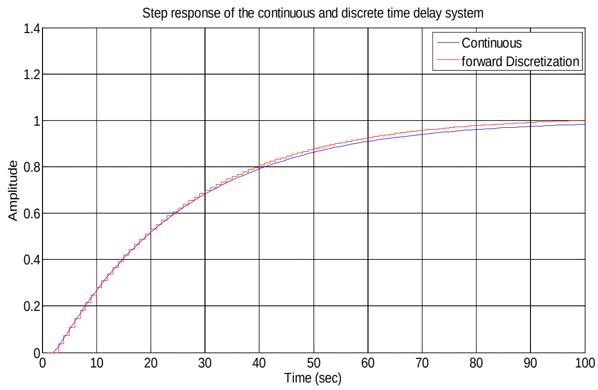

Fig. 1. Step response of the continuous and discrete time delay system.

This method is not recommended because the discrete-time equivalent may become unstable; which is commonly used in developing simple simulators, higher distortion with the forward rule.

Applying the method based on Backward difference method to the system considered in the state-space equations (29), and keeping the same sampling period T=0.2s. The following discrete time-delay system is obtained:

$$H(z)=C\left[\frac{I}{T}-A_0-A_1+\frac{A_1h}{T}-\left(\frac{A_1h}{T}+\frac{I}{T}\right)z^{-1}\right]^{-1}B \tag{31}$$

With:

$$A_0 = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} ; A_1 = \begin{pmatrix} 0 & 0 \\ -1 & 2 \end{pmatrix} ;$$

$$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} ; C = \begin{pmatrix} 1 & 1 \end{pmatrix} \ \ and \ \ h = qT = 0.2s$$

The simulation result in the step responses of the continuous system and approximate discrete system is depicted in figure2.

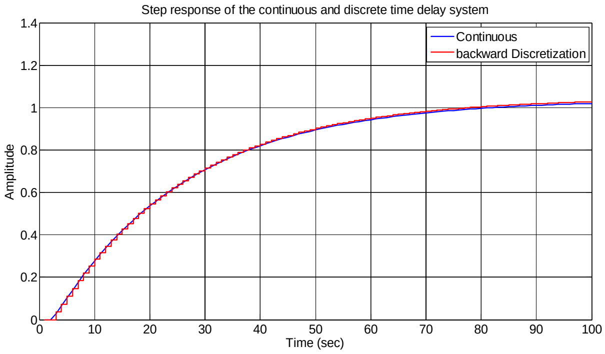

Fig. 2. Step response of the continuous and discrete time delay system.

The forward rectangular rule could cause a stable continuous filter to be mapped into an unstable digital filter, contrary to the rule of backward rectangular rule that keeps the system stable conditions except that the answer discrete time not following the continuous time.

Applying the method based on Bilinear z-transformation method to the system considered in the state-space equations (29), and keeping the same sampling period T=0.2s.The following discrete time-delay system is obtained:

$$H(z) = C(1+z^{-1})\left[I + A_1 h - A_0 \frac{T}{2} - A_1 \frac{T}{2} - (I - A_1 h + A_0 \frac{T}{2} + A_1 \frac{T}{2}) z^{-1} \right]^{-1} B \frac{T}{2} \quad (32)$$

With:

$$A_0 = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} ; A_1 = \begin{pmatrix} 0 & 0 \\ -1 & 2 \end{pmatrix} ;$$

$$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} ; C = \begin{pmatrix} 1 & 1 \end{pmatrix} \ \ and \ \ h = qT = 0.2s$$

The simulation result in the step responses of the continuous system and approximate discrete system is depicted in figure3.

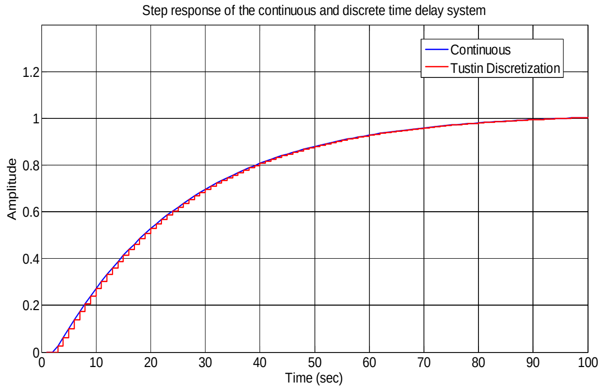

Fig. 3. Step response of the continuous and discrete time delay system.

It is clear that the value of the delay is shown by the simulation by the three discretization methods knowing that it did not affect the evolution of the system. The evolution of the response the field below presents a delay that does not degrade the performance of the system.

The bilinear transformation is motivated by considering the trapezoidal approximation of an integrator. It allows the passage of a Laplace transform in a z-transform. It uses the trapezoid method to calculate integral. This is a first-order approximation of the natural logarithm function that is an exact mapping of the z-plane to the P-plane. When the Laplace transform is performed on a discrete-time signal with each element of the discrete-time sequence attached to a correspondingly delayed unit impulse. The trapezoid rule maps the stable region in the P-plane exactly into the stable region of the z-plane. This method also offers the most accurate phase relative to the continuous system. It is clear that both digital controllers designed by emulation perform slightly worse than the continuous controller. As expected however, the trapezoidal integration method performs better than the forward and the backward rectangular method.

IV. COMPARISON BETWEEN EULER AND TUSTIN'S APPROXIMATIONS

We have discussed three different methods for obtaining discrete time approximations for continuous-time delay systems. Note how the discrete equivalent model predicts the continuous output at the sample instances. The facts that the second and third graph lines generate the same result show that the formula we have derived to calculate the discrete equivalent transfer function model is correct, Euler's backward differentiation method, which is commonly used in discretizing simple signal filters and industrial controllers. The backward differentiation method may give problems since it results in an implicit equation for the output variable, while the Forward differentiation method always gives an explicit equation. The Forward differentiation method is somewhat less accurate than the backward differentiation method, but it is simpler to use. In signal processing one is interested in making the difference between the original and the reconstructed signal as small as possible.

For continuous systems, the Laplace transform has played a major role. In fact, this linear transformation allows algebraically treat the linear operators, in the case of sampled systems; it has to introduce a transformation, called z, which has similar properties. The z-transfer function matrix will substitute for the p-transfer function matrix end of the Laplace transform. We can see very clearly the importance of the sampling period; a lower sampling period provides monitoring of the response of faithful continuous system with a longer period. The consequence of this is that the system with the lowest sampling period will be the one that will reach stability soon. So for the most accurate results possible relative to the continuous time it will be beneficial to not take too much sampling period. For better monitoring of the output in continuous time then choose a sampling period that will not be too great: but keep in mind that too little time require great precision the calculator. In signal processing literature the Tustin's method is frequently denoted the bilinear transformation method. The term bilinear is related to the fact that the imaginary axis in the complex P-plane for continuous-time systems is mapped or transformed onto the unity circle for the corresponding discrete-time system. In addition, the poles are transformed so that the stability property is preserved; this is not guaranteed in the Euler's Forward and Backward methods.

Tustin's method is the most accurate of these three methods, so I suggest it is the default choice. However, typically it is not much difference between the Tustin's method and the Euler's backward method.

The Euler's forward method is the least accurate method. Tustin's method and Euler's backward methods are implicit methods since the output variable, appears on both the left and the right side of the discretized expression, and it is in general necessary to solve the discretized expression for at each time-step, and this may be impractical.

V. CONCLUSION

In this paper we have proposed three methods for the discrete time approximation of continuous-time delay systems, using an integral numerical approximation, typically Euler's forward method, Euler's backward method or Tustin's method. These methods should be used when a continuous-time controller transfer function or filter transfer function are discretized to get an algorithm ready for programming. In such cases the input signal is a discrete-time signal with no holding. The discretization is based on some numerical method so that the behaviour of the discrete-time transfer function is similar to that of the continuous-time transfer function.

Usually we try to choose the sampling period an integer sub multiple of delay in addition to its forced choice for dynamic systems without delay. Finally it appears that the model using the Tustin's method is best suited for numerical simulation in the case of the continuous system is defined by a state model and in the case of a suitable choice of the sampling period. In a next job I am interested to study the discretization of these types of systems by the same methods taking into consideration the delay this time is manifold which appears in the state and control.

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